Spin structures, spinors and the Dirac operator: real versus complex manifolds

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Abstract. *We describe the relation between spin-structures, spinors and the Dirac operator on a (real) manifold and the analogous definitions in complex holomorphic terms. This may be useful for physicists interested in the algebraic geometric approach to superstrings.*

0. On a real manifold M, with a metric, there is a well known way to introduce spinor fields, as sections of a bundle associated to the spin bundle. This requires the definition of a spin structure (M is a spin-manifold) which can be done in a number of (equivalent) ways [cf. 1]. The Dirac operator acting on spinors is then of utmost importance for several areas of mathematics and physics.

In algebraic geometry, analogous objects have been defined for a long time too. On a complex manifold we speak about holomorphic square roots of the canonical bundle, sections of this line bundle, Dolbeault operator, etc..

To the best of the authors' knowledge the precise relation between these two languages, real and complex, has not been described in the literature in full detail. The relevant original references are [2-5], see also [6], [7] and [8]; some material is either known to the experts or belongs to the folklore. This note aims to present in a more systematic and complete way some facts and results on this topic, which may be of interest to physicists working in the area of superstring theory. It does not contain any new results.

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1. We assume that the Clifford algebra $\mathrm{Clif}(\mathbb{R}^n)$, the group $\mathrm{Spin}(n)$ and the double covering $\rho : \mathrm{Spin}(n) \to SO(n)$ are defined in the usual way, cf. for instance [7]. The complexified algebra $\mathrm{Clif}(\mathbb{R}^n) \otimes \mathbb{C} = \mathrm{Clif}(\mathbb{C}^n), n = 2m$ is a left $\mathrm{Spin}(n)$-module; it decomposes into 2^m simultaneous eigenspaces with respect to the right multiplication by $\alpha_j = ie_je_{j+m}, j = 1, \dots, m$, where $e_j, j = 1, \dots, n$ is the canonical orthonormal basis of $\mathbb{R}^n$. (Notice that $\alpha_j^2 = 1, [\alpha_j, \alpha_k] = 0$ and the right multiplication obviously commutes with the left multiplication by elements of $\mathrm{Spin}(n)$). These eigenspaces provide equivalent representations of $\mathrm{Spin}(n)$. Choose one of them, Δ, say the one corresponding to simultaneous eigenvalue -1 for each α_j. If furthermore decomposes as $\Delta = \Delta^+ \oplus \Delta^-$, according to the eigenvalues ± 1 or $\tau = i^m e_1 \dots e_n$ acting from the left, where $\Delta^\pm$ are now complex irreducible representations of $\mathrm{Spin}(n)$. Note that since $v\tau = -\tau v$ $(v \in \mathbb{C}^n)$ the Clifford multiplication by elements of $\mathbb{C}^n$ maps Δ^+ to Δ^- and vice versa.

In the sequel we shall need the group

$$\mathrm{Spin}_C(n) \equiv (\mathrm{Spin}(n) \times U(1))/\mathbf{Z}_2,$$

and the double covering

$$\begin{aligned} \rho_c : \mathrm{Spin}_C(n) &\to SO(n) \times U(1) \\ [g, \lambda] &\to (\rho(g), \lambda^2), \end{aligned}$$

where $\mathbb{Z}_2 = \{(1,1), (-1,-1)\}$ and the square bracket denotes the $\mathbb{Z}_2$-equivalence classes. It will prove useful that the homomorphism $h : U(m) \to SO(2m) \times U(1)$, $u \to (j(u), \det u)$ lifts to $\mathrm{Spin}_C(2m)$, i.e. there exists a homomorphism $\tilde{h} : U(m) \to \mathrm{Spin}_C(2m)$ such that $\rho_C \circ \tilde{h} = h$ (this is not the case for the natural imbedding $j : U(m) \to SO(2m)$). If we write $g \in U(m)$ in its diagonal form

(1) $$g = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_m})$$

(with respect to the basis $\tilde{e}_j$ of $\mathbb{C}^m$ such that $e_j = \tilde{e}_j, e_{j+m} = i\tilde{e}_j$ for $j = 1, \dots, m$), then $\tilde{h}$ is defined by

(2) $$\tilde{h}(g) \equiv \left[\prod_{j=1}^m (\cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2} e_je_{j+m}, \prod_{j=1}^m e^{\frac{i\theta_j}{2}}\right].$$

The representations $\Delta^\pm$ extend to representations $\Delta_C^\pm$ of $\mathrm{Spin}_C(n)$ in a natural way.

We introduce the standard notation $\Lambda^{0,+} = \oplus_{k \text{ even}} \Lambda^{0,k}$, $\Lambda^{0,-} = \oplus_{k \text{ odd}} \Lambda^{0,k}$, $\Lambda^{p,q} = \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1})$, $\Lambda^{1,0} = \mathrm{span}_C\{\beta_j\}$ and $\Lambda^{0,1} = \mathrm{span}_C\{\bar{\beta}_j\}$, where $\beta_j = e_j + ie_{j+m}, \bar{\beta}_j = e_j - ie_{j+m}$ for $j = 1, \dots, m$. Let $\gamma = \beta_1 \dots \beta_m$. The map

(3) $$\bar{\beta}_{i_1} \dots \bar{\beta}_{i_k}\gamma \to \sqrt{2}^k \bar{\beta}_{i_1} \wedge \dots \wedge \bar{\beta}_{i_k}$$

for $k = 0, \dots, m$ gives a canonical isomorphism between $\Delta_C^{\pm}$ and $\Lambda^{0,\pm}$. Among the 2^m linearly independent elements $\bar\beta_{i_1} \dots \bar\beta_{i_k}\gamma$, those with k even (resp. odd) form a basis of Δ^+ (resp. Δ^-) since on them right multiplication by α_j has for each j the eigenvalue -1 and left multiplication by τ has the eigenvalue $(-1)^k$.

The natural representation of $U(m)$ on $\Lambda^{0,\pm}$ under the isomorphism (3) is equivalent to the restriction of the representation $\Delta_C^{\pm}$ to $U(m)$. Namely, if we use the diagonal form (1) of $g \in U(m)$; then $\tilde h(g)\bar\beta_{i_1} \dots \bar\beta_{i_k}\gamma = e^{i\theta_{i_1}} \dots e^{i\theta_{i_k}}\bar\beta_{i_1} \dots \bar\beta_{i_k}\gamma$ which corresponds to the natural representation of $U(m)$ on $\Lambda^{0,k}$. Moreover, one can check that the isomorphism (3) intertwines the Clifford product in $\Delta_C^{\pm}$ and the multiplication $c : \mathbb{C}^n \otimes \Lambda^{0,\pm} \to \Lambda^{0,\mp}$, given by the formula $(v, \lambda) \to \frac{1}{\sqrt2}(v^{0,1} \wedge \lambda + v^{1,0}\lfloor\lambda)$. Here $v^{1,0}$ and $v^{0,1}$ are respectively the $\Lambda^{1,0}$ and $\Lambda^{0,1}$ parts of v and the exterior and interior multiplications are defined as follows:

$$e_k \wedge (e_{i_1} \wedge \dots \wedge e_{i_r}) = \begin{cases} (-1)^{s-1} e_{i_1} \wedge \dots \wedge e_{i_{s-1}} \wedge e_k \wedge e_{i_s} \wedge \dots \wedge e_{i_r} & \text{if } i_{s-1} < k < i_s \\ 0 & \text{otherwise} \end{cases}$$

$$e_k \lfloor (e_{i_1} \wedge \dots \wedge e_{i_r}) = \begin{cases} (-1)^{s-1} e_{i_1} \wedge \dots \wedge \hat e_{i_s} \wedge \dots \wedge e_{i_r} & \text{if } i_s = k \\ 0 & \text{otherwise} \end{cases}$$

(the hat denotes omission).

2. If we have a spin-structure $\mathrm{Spin}_g(M)$ on a real manifold M, with a metric g, we can construct the associated bundles: $TM := \mathrm{Spin}_g M \times_\rho \mathbf{R}^n$, $S^\pm := \mathrm{Spin}_g M \times_{\Delta^\pm} \Delta^\pm$. The Clifford multiplication by elements of $\mathbf{R}^n$ induces a map

$$\sigma : C^\infty(S^+ \otimes TM) \to C^\infty(S^-). \tag{4}$$

Then the following chain defines the (chiral part of the) Dirac operator $\not{D}\sigma \circ \nabla$

$$C^\infty(S^+) \xrightarrow{\nabla} C^\infty(S^+ \otimes T^*M) \equiv C^\infty(S^+ \otimes TM) \xrightarrow{\sigma} C^\infty S^-, \tag{5}$$

where ∇ is the covariant derivative (typically it comes from the Levi-Civita connection) and the tangent and cotangent bundles are identified via g.

Let now M be a complex manifold with a hermitian metric. In local coordinates $(z_1, \dots, z_m), z_j = x_j + iy_j, j = 1, \dots, m$ we denote $T'M = \mathrm{span}_C\{\frac{\partial}{\partial z_j}\}, T''M = \mathrm{span}_C\{\frac{\partial}{\partial \bar z_j}\}, \Lambda^{1,0}(M) = \mathrm{span}_C\{d z_j\}$ and $\Lambda^{0,1}(M) = \mathrm{span}_C\{d \bar z_j\}$. Moreover, let $\bar\partial : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M), (\bar\partial^2 = 0)$ and $\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M), (\partial^2 = 0)$ be the standard decomposition of the exterior derivative $d = \partial + \bar\partial$, where $\Lambda^{p,q}(M) = \Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M))$.

We have the following obvious

THEOREM 1. *A complex manifold M admits a canonical* spin_C*-structure obtained by lifting to* $\mathrm{Spin}_C(n)$ *via eq. (2) the* $U(m)$*-valued transition functions* $\varphi_{\alpha\beta}$ *of the bundle of hermitian frames on* M □

If we have a covariant (hermitian) derivative, then using the Clifford multiplication by vectors we can define a Dirac operator $\not{D}_C$ in analogy to (5). This operator can be thought of as a Dirac operator for spinors coupled to the $U(1)$-bundle associated to $\det : U(m) \to U(1)$ (see eq. (2)). It is possible to identify the bundle of such spinors with $\Lambda^{0,\pm}(M)$, by using the isomorphism (3) of the two representations $\Lambda^{0,\pm}$ and $\Delta_C^{\pm}$.

Moreover the symbol of $(\bar\partial + \bar\partial^*)$, where $\bar\partial^*$ is the formal adjoint of $\bar\partial$, exactly coincided under (3) with the symbol of $\not{D}_C$. Hence $\bar\partial + \bar\partial^*$ can differ from $\not{D}_C$ only by a 0-order term. From the fact that on a Kähler manifold there exists a coordinate system in which the first derivatives of the metric are zero at any fixed point follows that there does not exists any natural 0-order operator and hence:

THEOREM 2. *On a Kähler manifold*

$$\bar\partial + \bar\partial^* = \not{D}_C$$

□

3. When does a complex manifold admit a *usual* spin-structure? To answer this question observe that there are precisely two elements:

$$\pm\tilde h(g)(\det g)^{-\frac{1}{2}} \tag{6}$$

in $\mathrm{Spin}(n) \subset \mathrm{Spin}_C(n)$, which project to $g \in U(m)$. Choosing one of them is equivalent to choosing the sign of $\pm(\det g)^{-\frac{1}{2}}$. We can lift the transition functions $\varphi_{\alpha\beta}$ in $U(m)$, to $\mathrm{Spin}(n)$ (preserving the cocycle condition) if and only if we can choose consistently the square root of $\det(\varphi_{\alpha\beta}^{-1})$. This, in turn, means that there exists a square root L, $L \otimes L = K$, of the canonical line bundle $K = \Lambda^{m,0}(M)$ (whose transition functions, are $\det(\varphi_{\alpha\beta}^{-1})$). Moreover, there is a short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathcal{O}^* \xrightarrow{()^2} \mathcal{O}^* \to 1, \tag{7}$$

where $\mathcal{O}^*$ is the sheaf of germs of holomorphic functions, which leads to the long exact sequence

$$\begin{aligned} &H^0(M,\mathcal{O}^*) = \mathbb{C}^* \to H^1(M,\mathbb{Z}_2) \xrightarrow{\alpha} H^1(M,\mathcal{O}^*) \xrightarrow{2} \\ &H^1(M,\mathcal{O}^*) \xrightarrow{\beta} H^2(M,\mathbb{Z}_2) \to \dots \end{aligned} \tag{8}$$

The image of a holomorphic line bundle $L \in H^1(M, \mathcal{O}^*)$ under β is just its second Stiefel-Whitney class. Next, since γ is the square map, M is a spin-manifold if and only if K has a square root which is holomorphic. If M is compact, the first arrow of (8) maps to one and α is injective. Thus different spin-structures correspond to different holomorphic square roots of K. The previous arguments show the following

THEOREM 3. *A complex manifold M is a spin-manifold if and only if the canonical bundle K has a holomorphic square root. Moreover, if M is compact the inequivalent spin-structures are in a one-to-one correspondence with the holomorphic square roots of K.* □

Next, using the isomorphim (3), we can identify $\tilde{h}(u)(\det u)^{-\frac{1}{2}}$ with the operator $\Lambda^{0,\pm}u \otimes (\det u)^{-\frac{1}{2}}$ acting on $\Lambda^{0,\pm}$. Consequently the bundle of chiral spinors becomes $\Lambda^{0,\pm}(M) \otimes L$, where $L \otimes L = K$. Let us denote $\bar{\partial}_L$ the trivial extension of $\bar{\partial}$ on $\Lambda^{0,\pm}$ to $\Lambda^{0,\pm} \otimes L$ by $\bar{\partial}_L(\psi \otimes s) = \bar{\partial}\psi \otimes s$, where s is a holomorphic section of L.
We have:

THEOREM 4. *On a Kähler manifold*

$$\not{D} = \bar{\partial}_L + \bar{\partial}_L^*$$

Proof. we have

$$\not{D}(\psi \otimes s) = \sigma \circ \nabla(\psi \circ s) = (\sigma \circ \nabla\psi) \otimes s + \sigma(\psi \otimes \nabla s) =$$
$$= (\bar{\partial} + \bar{\partial}^*)\psi \otimes s = (\bar{\partial}_L + \bar{\partial}_L^*)(\psi \otimes s).$$

Here in the third equality we use theorem 2 and the fact that there is no (0, 1) part in ∇s (the connection is metric preserving). Thus the Clifford multiplication gives zero. In the last equality we use the fact that s is holomorphic and $\bar{\partial}^* s = 0$. □

4. We now want to see in the case of 1 complex dimension the special features of what has been previously discussed. In 2 real dimensions compact manifolds Σ without boundary are classified according to an integer g, the genus of the manifold. Moreover there is a one-to-one correspondence between complex structures and conformal structures. In fact given a conformal class of Riemannian metrics on Σ, there is a sufficiently fine open cover $\mathcal{U}_\alpha$ of Σ with coordinates (x_α, y_α) such that each metric G in this class has the form $G = e^{\varphi}(\mathrm{d}x_\alpha \otimes \mathrm{d}x_\alpha + \mathrm{d}y_\alpha \otimes \mathrm{d}y_\alpha) = e^{\varphi}(\mathrm{d}z_\alpha \otimes \mathrm{d}\bar{z}_\alpha)$ for some real-valued function φ on U_α, where $z_\alpha = x_\alpha + iy_\alpha$. The requirement that the metric is well defined on the overlaps implies that we have holomorphic (or antiholomorphic) transition function. Moreover we can identify, via the metric, holomorphic and antiholomorphic tensor fields.

Since dim $H^1(\Sigma, \mathbb{Z}_2) = 2^{2g}$ we have 2^{2g} spin-structures on Σ, known as theta characteristics. Now, it is easy to see that the bundle of positive chirality spinors $\Lambda^{0,0}(\Sigma) \otimes L = \Lambda^{0,+}(\Sigma) \otimes L$ get identified with L and the bundle $\Lambda^{0,1}(\Sigma) \otimes L = \Lambda^{0,-}(\Sigma) \otimes L$ of negative chirality spinors with L^{-1}, via the identification $d\bar{z} \otimes \sqrt{dz} \rightarrow \left(G\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)\right)^{-1} \sqrt{\frac{\partial}{\partial z}} = G^{z\bar{z}} \sqrt{\frac{\partial}{\partial z}}$ induced by the metric. Here L, as before, is an arbitrary but fixed square root of $\Lambda^{1,0}(\Sigma)$, and L^{-1} is the bundle dual to L (with inverse transition functions). The family of line-bundles forms and abelian group with respect to the tensor product. Denote $\tau^m \equiv \otimes^m L$, $\tau^0 = 1$, $\tau^{-m} = \otimes^m L^{-1}$ for $m > 0$, and $\Gamma(\tau^m)$ the space of section of τ^m. The operator $\not{D} = \bar{\partial} + \bar{\partial}^*$ becomes just $\bar{\partial}$ on L and $\bar{\partial}^*$ on L^{-1}. Hence $\bar{\partial}$ and $\bar{\partial}^*$ are the chiral parts of the Dirac operator in 1 complex dimension. As an operator from $\Gamma(\tau)$ to $\Gamma(\tau^{-1})$ $\not{D}$ acts as $\bar{\partial}$ times $G^{z\bar{z}}$. It extends trivially to give a family of operators $\not{D}_n : \Gamma(\tau^{n+2}) \rightarrow \Gamma(\tau^n)$. The formal adjoint will be $\not{D}_n^* : \Gamma(\tau^n) \rightarrow \Gamma(\tau^{n+2}), \psi \rightarrow -(G^{z\bar{z}})^{-n/2} \partial((G^{z\bar{z}})^{n/2} \psi)$. From the Riemann-Roch theorem we obtain that the index of this operator is $(n+1)(g-1)$. This theorem is a prototype of more general index theorems [6].

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